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NDIM achievements: massive, arbitrary tensor rank and N-loop insertions in Feynman integrals

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Abstract. One of the main difficulties in studying quantum field theory, in the perturbative regime, is the calculation of D-dimensional Feynman integrals. In general, one introduces the so-called Feynman parameters and, associated with them, the cumbersome parametric integrals. Solving these integrals beyond the one-loop level can be a difficult task. The negative-dimensional integration method (NDIM) is a technique whereby such a problem is dramatically reduced. We present the calculation of two-loop integrals in three different cases: scalar ones with three different masses, massless with arbitrary tensor rank, with and N insertions of a two-loop diagram.

1. Introduction

Perturbative calculations in quantum field theory are often a hard task, especially if one does not have a suitable approach to tackle the problem. Of the several techniques available the most popular is Feynman parametrization [1]; in fact, if one is clever enough, hefty calculations (at the two-loop level) can be performed [2]. However, in our view, this is not the most effective or elegant method to solve Feynman integrals, whether one is considering covariant or non-covariant gauges.

On the other hand, Chetyrkin *et al* developed integration by parts in configuration space (associated with Gegenbauer polynomials) and performed (at four loops) even heftier calculations [3]. However, their technique has a drawback: if the diagrams have more than two external legs, manipulation of the Gegenbauer polynomials becomes very difficult to handle [2].

Mellin–Barnes contour integration is a third possible approach. Each propagator is Mellin-transformed [4]—a very simple step—and using Barnes' lemmas and summing over the residues, it is possible to write the Feynman integrals as hypergeometric functions or hypergeometric-like series. Smirnov [5] solved the massless double-box with the Mellin–Barnes approach. Such Mellin integrals are parametric-like integrals—although they are much simpler to solve than Feynman-like ones because Cauchy's theorem can be applied straightforwardly.

The negative-dimensional integration method (NDIM) [6] is a technique in which the parametric integrals do not appear. We start with a Gaussian integral, which is well behaved, perform a Taylor expansion and solve systems of algebraic equations. All the calculations can be done analytically and the results are given for arbitrary exponents of propagators and space–time dimension D, as in the standard dimensional regularization [7]. Even integrals pertaining to the trickier non-covariant gauges, such as the light-cone [8] and Coulomb [9]

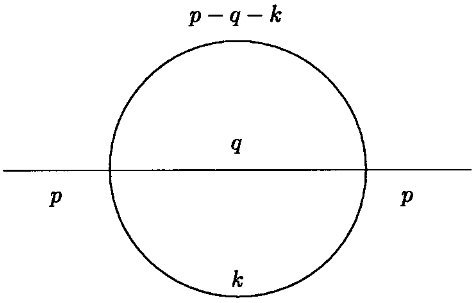

Figure 1. Simplest two-loop massless Feynman diagram: setting sun.

ones, can be performed using the same approach (however, we shall deal with them in a future work). In the following sections we show how this can be done.

The outline of our paper is as follows: in section 2 we consider the simplest two-loop diagram, our workhorse, as a pedagogical example and present, in a clean way, the technique of negative-dimensional integration. Section 3 is devoted to the same diagram but now with tensorial structure. The NDIM can handle vector, second-rank tensor and higher-order integrals, all at the same time. In section 4 we replace massless propagators by massive ones, and in section 5, we consider the massless diagram with N insertions of the same type. In section 6, we present our concluding remarks.

2. Simplest two-loop diagram

To make things clear we begin with the diagram of figure 1. In a massless scalar theory it is represented by

$$A = \int \frac{\mathrm{d}^D q\, \mathrm{d}^D k}{(q^2)(k^2)(p-q-k)^2}. \tag{1}$$

2.1. Negative-dimensional approach

Consider the Gaussian integral,

$$G = \int \mathrm{d}^D q\, \mathrm{d}^D k \exp\left[-\alpha q^2 - \beta k^2 - \gamma (p-q-k)^2\right] \tag{2}$$

where (α, β, γ) are such that G is well behaved. We will see that it is the generating functional of negative-dimensional integrals, see (5). Integrating over q and k is very easy:

$$G = \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left(-\frac{\alpha\beta\gamma}{\lambda} p^2\right) \tag{3}$$

where $\lambda = \alpha\beta + \alpha\gamma + \beta\gamma$. Expanding (3) in Taylor series and using a multinomial expansion for λ we get

$$G = \pi^D \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{\alpha^{n_{123}} \beta^{n_{124}} \gamma^{n_{134}} (p^2)^{n_1}}{n_1! n_2! n_3! n_4!} (-n_1 - D/2)! \tag{4}$$

where, due to our multinomial expansion, $n_{234} = -n_1 - D/2$, and we define

$$n_{12} = n_1 + n_2 \qquad n_{123} = n_1 + n_2 + n_3$$

and so forth.

On the other hand, Taylor expanding (2),

$$G = \sum_{i,j,l=0}^{\infty} \frac{(-1)^{i+j+l}\alpha^i \beta^j \gamma^l}{i!j!l!} \int \mathrm{d}^D q \, \mathrm{d}^D k (q^2)^i (k^2)^j (p-q-k)^{2l} \tag{5}$$

one generates the negative-D integral. Now, comparing (4) and (5) we solve for the integral above:

$$\mathcal{A}(i,j,l) = \int \mathrm{d}^D q \mathrm{d}^D k (q^2)^i (k^2)^j (p-q-k)^{2l} \tag{6}$$

$$= \frac{\pi^D i!j!l!}{(-1)^{i+j+l}} \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{(p^2)^{n_1}(-n_{1234} - D/2)!}{n_1!n_2!n_3!n_4!} \delta_{n_{123},i}\delta_{n_{124},j}\delta_{n_{134},l}\delta_{n_{234},-n_1-D/2} \tag{7}$$

where the Kronecker deltas give rise to a system of four equations and four 'unknowns'. Plugging the solution into (7) provides us the result of $\mathcal{A}(i,j,l)$, namely

$$\mathcal{A}(i,j,l) = \frac{\pi^D \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1-\sigma-D/2)}{\Gamma(1-i-D/2)\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1+\sigma)} (p^2)^{\sigma} \tag{8}$$

where $\sigma = i+j+l+D$. However, the above result is valid on the negative-dimensional region and positive exponents of propagators $(i,j,l \geqslant 0)$. To bring it to our real physical world we must invoke the principle of analytic continuation. This is a quite simple step [10]: we group the gamma functions into Pochhammer symbols and use one of its properties,

$$(a)_k \equiv (a|k) = \frac{\Gamma(a+k)}{\Gamma(k)} \qquad (a|-k) = \frac{(-1)^k}{(1-a|k)}. \tag{9}$$

Doing so, we get the result in our positive-dimensional world,

$$\mathcal{A}^{AC}(i,j,l) = \pi^D (p^2)^{\sigma} (-i|i+j+D/2) \times (-j|j+k+D/2)(-k|i+k+D/2)(\sigma+D/2|-2\sigma-D/2) \tag{10}$$

and negative exponents of propagators $(i,j,l) \leqslant 0$ in Euclidean space. Observe that the result is symmetric in the propagators exponents reflecting the symmetry of figure 1.

The recipe for calculating Feynman integrals using NDIM technology is quite simple: (i) to each loop write a Gaussian integral whose arguments are the propagators of the diagram in question; (ii) complete the square(s) and integrate; (iii) take the original Gaussian integral, Taylor expand the exponential and change the order $\sum \leftrightarrow \int$: this operation generates negative-dimensional integrals; (iv) the equality of these two expressions must hold, so one can solve for the negative-dimensional integral and gets n-fold series involving Kronecker deltas; (v) such Kronecker deltas give rise to a system of linear algebraic equations: in most cases this does not have a unique solution, since it is a rectangular matrix [11]; (vi) plugging the solution(s) into that series representation provides the result of the negative-dimensional integral—sometimes in massless cases there appear degenerate solutions [10]; (vii) analytically continue the referred result to the positive-dimensional region using the above property of Pochhammer symbols. The whole procedure is quite simple, and we will show it for cases of interest.

3. Tensorial structure

The previous result was obtained with amazing ease: arbitrary negative exponents of propagators, positive dimension and no numerical calculations. The reader can rightfully ask: Does NDIM work for tensorial numerators as well? The answer is yes [12]. We need to modify only one thing.

Consider the integral,

$$\mathcal{B}(i,j,l,m) = \int \mathrm{d}^D q\, \mathrm{d}^D k (q^2)^i (k^2)^j (p-q-k)^{2l} (2q\cdot p)^m \tag{11}$$

from

$$\begin{aligned} G_T &= \int \mathrm{d}^D q\, \mathrm{d}^D k \exp\left[-\alpha q^2 - \beta k^2 - \gamma(p-q-k)^2 - 2\phi q\cdot p\right] \\ &= \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left(-\frac{\alpha\beta\gamma + 2\beta\gamma\phi - \beta\phi^2 - \gamma\phi^2}{\lambda} p^2\right) \\ &= \pi^D \sum_{n_1,\ldots,n_7=0}^{\infty} \frac{(-1)^{n_{12}} 2^{n_2} \alpha^{n_{157}} \beta^{n_{12356}} \gamma^{n_{12467}} \phi^{n_2+2n_{34}} (p^2)^{n_{1234}}}{n_1! n_2! n_3! n_4! n_5! n_6! n_7!} \\ &\quad \times (-n_{1234} - D/2)! \\ &= \sum_{i,j,l,m=0}^{\infty} \frac{(-1)^{i+j+l+m} \alpha^i \beta^j \gamma^l \phi^m}{i! j! l! m!} \\ &\quad \times \int \mathrm{d}^D q;\ \mathrm{d}^D k (q^2)^i (k^2)^j (p-q-k)^{2l} (2q\cdot p)^m. \end{aligned} \tag{12}$$

As we did in the previous section, solving for $\mathcal{B}$ leads to

$$\begin{aligned} \mathcal{B}(i,j,l,m) = \frac{\pi^D i! j! l! m!}{(-1)^{i+j+l+m}} \sum_{n_1,\ldots,n_7=0}^{\infty} \frac{2^{n_2} (-1)^{n_{12}} (p^2)^{n_{1234}} (-n_{1234} - D/2)!}{n_1! n_2! n_3! n_4! n_5! n_6! n_7!} \\ \times \delta_{n_{157},i} \delta_{n_{12356},j} \delta_{n_{12467},l} \delta_{n_{n_2+2n_{34}},m} \delta_{n_{567},-n_{1234}-D/2}. \end{aligned} \tag{13}$$

Observe that now the system generated by the Kronecker deltas does not have a unique solution, since there are seven 'unknowns' and five equations and, consequently, two of them will be left undetermined. There are $C_5^7 (=7!/5!2! = 21)$ distinct ways of solving this 5×7 system, but five of them have no solution at all.

In a previous work [12] we showed that all non-trivial solutions are legitimate and lead to the same result for the Feynman integral in question. Here the same occurs—but we will not prove it.

Note that the tensorial sector of the Feynman integral $\mathcal{B}(i,j,l,m)$ is contained in the factor $(2q\cdot p)^m$ and for this very reason the exponent m cannot be analytically continued to allow for negative values. In other words, we must invoke the principle of analytic continuation for three exponents of propagators leaving the fourth, m, untouched.

All the solutions will give rise to a double series of hypergeometric type, since we have a 7-fold series and only five Kronecker deltas in (13). However, from the theory of hypergeometric series [13] we know that when one of its numerator parameters is a negative integer, say $-m$, the series is truncated and has only m terms. For this reason we will consider a solution that is obtained when $\{n_3, n_4\}$ are left undetermined:

$$\mathcal{B}(i,j,l,m) = g_{\mathcal{B}} \sum_{n_3,n_4=0}^{\infty} \frac{(-m/2|n_{34})(1/2 - m/2|n_{34})(D/2 + j|n_4)(D/2 + l|n_3)}{(1+\sigma' - m|n_{34})(1 - i - m - D/2|n_{34}) n_3! n_4!} \tag{14}$$

where

$$g_{\mathcal{B}} = \frac{(-\pi)^D (p^2)^{\sigma'} 2^m \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1-\sigma'-D/2)}{\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1-i-m-D/2)\Gamma(1+\sigma'-m)}. \tag{15}$$

$\sigma' = \sigma + m$ and we use the relation [13]

$$(a|2b) = 2^{2b}(a/2|b)(1/2 + a/2|b).$$

Note that for positive m, which is the relevant condition, the series (14) is always truncated (when it is even, the first factor in the numerator is the responsible, whereas for m-odd the second factor truncates the series). Analytically, continuation of $g_{\mathcal{B}}$ gives

$$g_{\mathcal{B}}^{AC} = \pi^D (p^2)^{\sigma'} 2^m (-i|-j-l-D)(-j|j+l+D/2)$$
$$\times(-l|j+l+D/2)(\sigma'+D/2|i+m-\sigma') \tag{16}$$

and the final result, in positive dimension, is given by the series in equation (14) times $g_{\mathcal{B}}^{AC}$:

$$\mathcal{B}^{AC}(i,j,l,m) = g_{\mathcal{B}}^{AC} \sum_{n_3,n_4=0}^{\infty} \frac{(-m/2|n_{34})(1/2-m/2|n_{34})(D/2+j|n_4)(D/2+l|n_3)}{(1+\sigma'-m|n_{34})(1-i-m-D/2|n_{34})n_3!n_4!}. \tag{17}$$

Observe that for $m = 0$ we obtain the scalar case (10); for $m = 1$ an integral with vector numerator; for $m = 2$ second rank tensor, and so forth. The results, of course, are contracted with the external momentum p^μ. The astonishing point is that all these *new* results are contained *in the same formula*, namely equation (17).

4. Massive propagators

The NDIM is a powerful technique. It gives, simultaneously, vector, second-rank tensor and higher-order integrals. A second question one could ask is: Does NDIM work for massive propagators as well? The answer is also yes and we need to make only slight modifications.

Let our generating function, corresponding to the diagram of figure 1 but where now the virtual particles have distinct masses, be

$$G_m = \int \mathrm{d}^D q\, \mathrm{d}^D r \exp\{-\alpha(q^2-m_1^2) - \beta(r^2-m_2^2) - \gamma[(p-q-r)^2-m_3^2]\}$$
$$= \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\alpha^i\beta^j\gamma^k}{i!j!k!}\mathcal{M}(i,j,k) \tag{18}$$

where

$$\mathcal{M}(i,j,k) = \int \mathrm{d}^D q\, \mathrm{d}^D r (q^2-m_1^2)^i (r^2-m_2^2)^j [(p-q-r)^2-m_3^2]^k. \tag{19}$$

Using (3) we get

$$G_m = \exp(\alpha m_1^2 + \beta m_2^2 + \gamma m_3^2) G \tag{20}$$

and following the general procedure, as in the previous cases, one can write the integral as

$$\mathcal{M}(i,j,k) = \frac{\pi^D i!j!k!}{(-1)^{i+j+k}} \sum_{n_1,\ldots,n_7=0}^{\infty} \frac{(-n_4-D/2)!(m_1^2)^{n_1}(m_2^2)^{n_2}(m_3^2)^{n_3}(-p^2)^{n_4}}{n_1!\ldots n_7!}$$
$$\times\delta_{n_{1456},i}\delta_{n_{2457},j}\delta_{n_{3467},k}\delta_{n_{4567},-D/2}. \tag{21}$$

In this case, the Kronecker deltas give rise to a 4×7 system of linear algebraic equations. We have 35 possible solutions for such a system but 15 of them have no solution at all. So, we are left with 20 triple series, following the prescription of summing the ones which have the same variables [8, 11] (this is equivalent to saying that we sum those which have the same region of convergence [10]), we get four possible triple series (of hypergeometric type) representing the Feynman integral $\mathcal{M}(i,j,k)$:

$$\left(\frac{m_1^2}{p^2}\right)^a \left(\frac{m_2^2}{p^2}\right)^b \left(\frac{m_3^2}{p^2}\right)^c \qquad \left(\frac{m_1^2}{m_3^2}\right)^a \left(\frac{m_2^2}{m_3^2}\right)^b \left(\frac{p^2}{m_3^2}\right)^c$$
$$\left(\frac{m_1^2}{m_2^2}\right)^a \left(\frac{m_3^2}{m_2^2}\right)^b \left(\frac{p^2}{m_2^2}\right)^c \qquad \left(\frac{m_2^2}{m_1^2}\right)^a \left(\frac{m_3^2}{m_1^2}\right)^b \left(\frac{p^2}{m_1^2}\right)^c.$$

For the first one we have eight solutions and the second, third and fourth have four, so we have 20 possible solutions of the system generated by the Kronecker deltas ($8 + 4 + 4 + 4 = 20$). Since the last three have the same form, due to the symmetry of the diagram in question, we will consider only one of them; the others can be obtained changing masses and exponents of propagators.

We quote only the results, where the analytic continuation process has been already carried out. The first triple series is

$$\begin{aligned}\mathcal{M}_1(i,j,k,\{z\}) = {} & [f_1\mathcal{F}_C^{(3)}(-k, 1-k-D/2; 1+i+D/2, 1+j+D/2, 1-k-D/2) \\ & +(j\leftrightarrow k)+(i\leftrightarrow k)] + [f_2\mathcal{F}_C^{(3)}(-j-k-D/2, 1-j-k-D; 1+i \\ & +D/2, 1-j-D/2, 1-k-D/2) + (i\leftrightarrow j) + (k\leftrightarrow i)] \\ & + f_3\mathcal{F}_C^{(3)}(-\sigma, 1-\sigma-D/2; 1-i-D/2, 1-j-D/2, 1-k-D/2) \end{aligned} \tag{22}$$

where

$$f_1 = (-\pi)^D(m_1^2)^{i+D/2}(m_2^2)^{j+D/2}(p^2)^k(-i|-D/2)(-j|-D/2)$$

$$f_2 = (-\pi)^D(m_1^2)^{i+D/2}(-p^2)^{j+k+D/2}(-i|-D/2)(-j|-k-D/2) \times(j+k+D|-k-D/2)(-k|2k+D/2)$$

$$f_3 = (-\pi)^D(p^2)^{\sigma}(-i|2i+D/2)(-j|2j+D/2)(-k|2k+D/2)(D/2|-\sigma-D/2).$$

Note that one of the solutions has a factor $(0|D/2) = \Gamma(D/2)/\Gamma(0)$, so that we have in fact seven terms (in positive dimension) instead of eight (in negative dimension). The second triple series of hypergeometric type is represented by a sum of four terms:

$$\begin{aligned}\mathcal{M}_2(i,j,k,\{w\}) = {} & g_1\mathcal{F}_C^{(3)}(-k, D/2; 1+i+D/2, 1+j+D/2, D/2) \\ & +g_2\mathcal{F}_C^{(3)}(-j-k-D/2, -j; 1-j-D/2, D/2, 1+i+D/2) \\ & +g_3\mathcal{F}_C^{(3)}(-i, -i-k-D/2; 1-i-D/2, 1+j+D/2, D/2) \\ & +g_4\mathcal{F}_C^{(3)}(-\sigma, -i-j-D/2; 1-i-D/2, 1-j-D/2, D/2) \end{aligned} \tag{23}$$

where the hypergeometric series which appears in the above results

$$\mathcal{F}_C^{(3)}(\alpha,\beta;\gamma,\theta,\phi,\{x\}) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha|n_{123})(\beta|n_{123})x_1^{n_1}x_2^{n_2}x_3^{n_3}}{n_1!n_2!n_3!(\gamma|n_1)(\theta|n_2)(\phi|n_3)} \tag{24}$$

is a Lauricella function [14–16] which converges if

$$|x_i| < 1 \qquad \text{and} \qquad \sqrt{|x_1|}+\sqrt{|x_2|}+\sqrt{|x_3|} < 1 \tag{25}$$

and we define

$$g_1 = \frac{(-\pi)^D}{(-1)^{i+j+k}}(m_1^2)^{i+D/2}(m_2^2)^{j+D/2}(m_3^2)^k(-i|-D/2)(-j|-D/2)$$

$$g_2 = \frac{(-\pi)^D}{(-1)^{i+j+k}}(m_1^2)^{i+D/2}(m_3^2)^{j+k+D/2}(D/2|j)(-i|-D/2)(-k|-j-D/2)$$

$$g_3 = \frac{(-\pi)^D}{(-1)^{i+j+k}}(m_2^2)^{j+D/2}(m_3^2)^{i+k+D/2}(-j|-D/2)(-k|-i-D/2)(D/2|i)$$

$$g_4 = \pi^D(-m_3^2)^{\sigma}(-i|i+j+D/2)(-j|i+j+D/2) \times(D/2|-i-j-D)(-k|-i-j-D).$$

The variables in $\mathcal{M}_1$ and $\mathcal{M}_2$ are respectively $\{x\} = \{z\}$ and $\{x\} = \{w\}$, where

$$z_1 = m_1^2/p^2 \qquad z_2 = m_2^2/p^2 \qquad z_3 = m_3^2/p^2$$
$$w_1 = m_1^2/m_3^2 \qquad w_2 = m_2^2/m_3^2 \qquad w_3 = p^2/m_3^2.$$

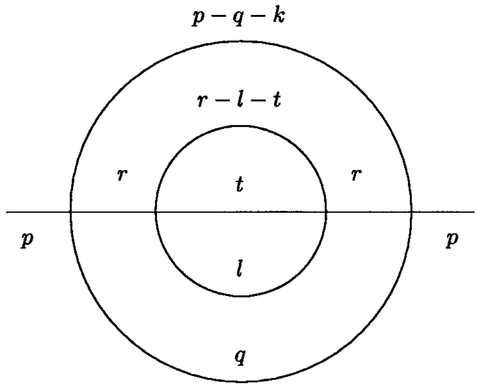

Figure 2. Four-loop massless Feynman diagram: setting sun with one insertion.

Observe that our results, which were obtained simultaneously, agree with those of Berends *et al* [14] (which were obtained by analytic continuation). In their work, $\mathcal{M}_2(i, j, k, \{w\})$ was calculated first and then analytically continued to allow other values of momenta and masses, resulting in $\mathcal{M}_1(i, j, k, \{z\})$. However, if the analytic continuation formula is not known the result in the other region is difficult to obtain. On the other hand, the NDIM provides *all* the results simultaneously, allowing us to also obtain new analytic continuation formulae [8, 11].

5. Insertions

A third question the reader could pose is: Does the NDIM also accomplish good results for higher numbers of loops? Our main objective is to apply the NDIM to higher loops, but up to now we have only been able to 'glue' diagrams with two external legs.

Consider the diagram of figure 2. It has four loops and is represented by

$$\mathcal{C}(i, j, k, m, n, s) = \int \mathrm{d}^D q\, \mathrm{d}^D r\, \mathrm{d}^D l\, \mathrm{d}^D t\, (q^2)^i (r^2)^j (p - q - r)^{2k} (l^2)^m (t^2)^n (r - l - t)^{2s} \tag{26}$$

in a massless theory. Observe that the integrals in l and t are equal, see (6), to $\mathcal{A}(m, n, s;\, p^2 \to r^2)$. Rewriting, we get

$$\begin{aligned}\mathcal{C}(i, j, k, m, n, s) &= \int \mathrm{d}^D q\, \mathrm{d}^D r (q^2)^i (r^2)^j (p - q - r)^{2k} \mathcal{A}(m, n, s;\, p^2 \to r^2)\\ &= \pi^D (-m|m + n + D/2)(-n|n + s + D/2)(-s|m + s + D/2)\\ &\quad \times (\sigma_1 + D/2| - 2\sigma_1 - D/2)\mathcal{A}(i, j + \sigma_1, k)\\ &= \pi^{2D} (p^2)^{\sigma_1+\sigma_2} (-i|i + j + \sigma_1 + D/2)(-j - \sigma_1|j + \sigma_1 + k + D/2)\\ &\quad \times (-k|i + k + D/2)(\sigma_2 + D/2| - 2\sigma_2 - D/2)(-m|m + n + D/2)\\ &\quad \times (-n|n + s + D/2)(-s|m + s + D/2)(\sigma_1 + D/2| - 2\sigma_1 - D/2)\end{aligned} \tag{27}$$

where $\sigma_1 = m + n + s + D$, $\sigma_2 = i + j + k + D$.

Let us study the diagram of figure 3, namely the one that has a single propagator replaced N times by the graph of figure 1:

$$\mathcal{C}_N(\nu_1, \nu_2, \ldots, \nu_{3N}) = \int \ldots \int \prod_{i=1}^{i=N} \mathrm{d}^D q_i\, \mathrm{d}^D r_i\, (q_i^2)^{\nu_{3i-2}} (r_i^2)^{\nu_{3i-1}} [(r_{i-1} - q_i - r_i)^2]^{\nu_{3i}} \tag{28}$$

where $r_0^\mu = p^\mu$ is the external momentum and N is the number of insertions. When $N = 1$ the above integral reduces to $\mathcal{A}(\nu_1, \nu_2, \nu_3)$.

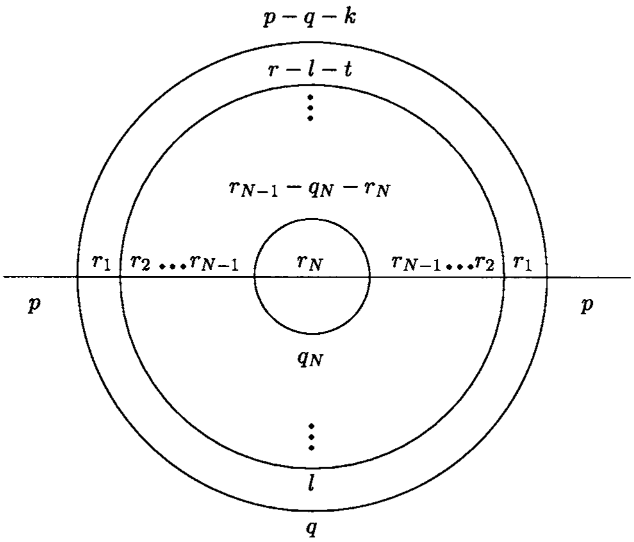

Figure 3. $2N$-loop massless Feynman diagram: setting sun with N insertions of the setting sun graph.

Applying the above procedure we can easily solve such a scalar integral:

$$\mathcal{C}_N^{AC}(\nu_1, \nu_2, \ldots, \nu_{3N}) = \mathcal{A}^{AC}(\nu_1, \nu_2, \nu_3)\mathcal{A}^{AC}(\nu_4, \nu_5 + \sigma_1, \nu_6) \times \cdots \times \mathcal{A}^{AC}(\nu_{3N-2}, \nu_{3N-1} + \sigma_1 + \cdots + \sigma_{N-1}, \nu_{3N}) \tag{29}$$

where $\sigma_N = \nu_{3N-2} + \nu_{3N-1} + \nu_{3N} + D$.

5.1. *Water melon diagram. Massless case*

Recently, the water melon diagram was considered in the massive case, whether in four dimensions and of analytic result [14] or integral representations of it suitable for numerical calculations [16].

Applying the NDIM we can solve, exactly, such water melon diagrams, see figure 4. Here, we will consider the scalar massless case. At the two-loop level the appropriate water melon is the graph of figure 1, our setting sun diagram. At the three-loop level we begin with

$$G_{WM} = \int \mathrm{d}^D q\, \mathrm{d}^D r\, \mathrm{d}^D k \exp\left[-\alpha q^2 - \beta r^2 - \gamma k^2 - \theta(p-q-r-k)^2\right] = \left(\frac{\pi^3}{\zeta}\right)^{D/2} \exp\left(-\frac{\alpha\beta\gamma\theta}{\zeta}p^2\right) \tag{30}$$

where $\zeta = \alpha\beta\gamma + \alpha\beta\theta + \alpha\gamma\theta + \beta\gamma\theta$. After a little of algebra we get a 4×4 system which gives

$$\mathcal{W}_3(i,j,l,m) = \int \mathrm{d}^D q\, \mathrm{d}^D r\, \mathrm{d}^D k (q^2)^i (r^2)^j (k^2)^l (p-q-r-k)^{2m} = (\pi^{3D/2} i!j!l!m!\Gamma(1-\rho_3-D/2)(p^2)^{\rho_3})\{(-1)^{i+j+l+m}\Gamma(1-i-D/2) \times\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1-m-D/2)\Gamma(1+\rho_3)\}^{-1} \tag{31}$$

in negative dimension and Euclidean space. We define $\rho_3 = i+j+l+m+3D/2$. Generalizing

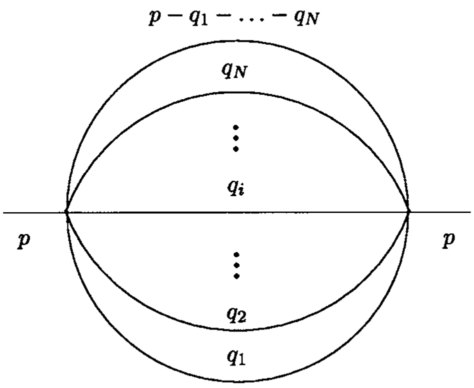

Figure 4. Scalar massless water melon Feynman diagram with N-loops.

this result to N-loops is quite easy:

$$\mathcal{W}_N(\{\nu_n\}) = \int \ldots \int \prod_{i=1}^{i=N} \mathrm{d}^D q_i (q_i^2)^{\nu_i} (p - q_1 - q_2 - \cdots - q_N)^{2\nu_{N+1}}$$

$$= \frac{\pi^{ND/2} \nu_1! \cdots \nu_{N+1}! \Gamma(1 - \rho_N - D/2)(p^2)^{\rho_N}}{(-1)^{\Sigma\nu} \Gamma(1 - \nu_1 - D/2) \cdots \Gamma(1 - \nu_{N+1} - D/2) \Gamma(1 + \rho_N)} \qquad (32)$$

where $\Sigma\nu = \nu_1 + \cdots + \nu_{N+1}$ and $\rho_N = \Sigma\nu + ND/2$. Carrying out the analytic continuation we get

$$\mathcal{W}_N^{AC}(\{\nu_n\}) = \pi^{ND/2}(p^2)^{\rho_N}(-\nu_1|2\nu_1 + D/2)$$

$$\times(-\nu_2|2\nu_2 + D/2) \cdots (-\nu_N|2\nu_N + D/2)(\rho_N + D/2| - 2\rho_N - D/2) \qquad (33)$$

as the result for negative exponents of propagators and positive dimension. As far as we know, this result was not known for arbitrary exponents of propagators. Observe that when one is allowed to 'gluing' diagrams, the result can easily be generalized to N-loops.

6. Conclusion

The NDIM is a suitable technique to tackle the task of calculating multiloop Feynman integrals. Massless, massive, scalar, tensorial—even the ones for non-covariant gauges, such as the light-cone gauge [8, 9], are easily performed. In all of them, exponents of propagators are left arbitrary as well as the dimension D, just as in plain dimensional regularization. The usual parametric integrals are replaced by one Gaussian integral over each momentum flowing in the loop, and the main task is to solve systems of linear algebraic equations. Another point that we would like to stress is that no numerical calculations are required at all.

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